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RELATIVISTIC CHARGED FLUID FLOW II: GENERALIZED LARMOR AND HELMHOLTZ THEOREMS

BY

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ABSTRACT

The covariant Larmor theorem is derived within the framework of Special Relativity for an inviscid charged fluid in the presence of an arbitrary electromagnetic field and a gravitational field which is described by a simple scalar potential. The flow need not be adiabatic and the fluid need not be barotropic. The covariant form of the Larmor theorem, which is an antisymmetric tensor equation, is equivalent to two 3-vector equations, one of which is the fluid equation of motion. The other is just the familiar 3-vector statement of the Larmor theorem except for an extra precession that is thermal in origin. The relativistic Helmholtz equation is derived from the covariant Larmor theorem. The vorticity is defined as the curl of the usual canonical momentum of a charged particle in an electromagnetic field except that the particle rest mass is a variable that includes contributions that are proportional to the gravitational potential and the specific enthalpy of the fluid. For this definition of the vorticity it is found that the vorticity flux diffuses through the fluid except when the flow is isentropic. It is shown, however, that when an appropriate thermal 4-potential is included in the definition of the canonical momentum, the vorticity flux remains frozen in the fluid even when viscosity is present and the flow is nonadiabatic. A microscopic interpretation of this generalized Helmholtz equation is given.

CONTENTS

RELATIVISTIC CHARGED FLUID FLOW

II: Generalized Larmor and Helmholtz Theorems

I. INTRODUCTION

In the preceding paper,* henceforth referred to as I, it was shown that the entropy-dependent force terms in the relativistic equation of motion could be expressed in terms of an antisymmetric entropy force tensor whose dynamical role is analogous to that of the electromagnetic field tensor. This analogy is most clearly evident in the relativistic statement of the Larmor theorem for a charged inviscid fluid, which is derived in Section II of the present paper.

Use of the entropy force tensor also makes possible the derivation of the relativistic generalization of the Helmholtz equation for the time dependence of the fluid vorticity. This is carried out in Section III. In the form of the Helmholtz equation derived in Section III, the vorticity is defined as the curl of the canonical particle 4-momentum $\mu v^j + (q/c)A^j$ where μ is the particle rest mass, including contributions from the gravitational and thermal energy, q is the particle charge, and A^j is the electromagnetic 4-vector potential in Gaussian units. As in I, v^j denotes the particle 4-velocity and c, of course, is the speed of light.

The form of the Helmholtz equation in terms of a vorticity defined as the curl of $\mu v^j + (q/c) A^j$ has the disadvantage that the vorticity so defined is in general not simply carried along with the fluid flow, but rather diffuses across

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flow lines. In Section IV, however, it is shown that it is possible to define the vorticity in such a way that no diffusion occurs. The vorticity having this property is defined as the curl of a generalized canonical momentum that includes not only the contribution $(q/c)A^j$ from the electromagnetic field, but also an analogous contribution that accounts for viscosity as well as for the entropy-dependent forces.

Notation

The notation employed will be the same as that used in I with the exception of a modification in the notation for temperature. Inasmuch as we have committed ourselves to a 4-vector description of temperature, it would be desirable to introduce a notation that is analogous to that for particle mass and density. In the fluid rest frame these are designated by m and ρ respectively, and in the observer's frame by m^* and ρ^*. Thus temperature in the fluid rest frame will be designated by T (instead of $\overset{\circ}{T}$ as in I), and in the observer's frame by T^* (instead of T). Equation (3.12) of I will now be written

$$T^j = T v^j/c = T^*(1, \mathbf{v}/c) \tag{1.1a}$$

where

$$T^* = \Gamma T, \tag{1.1b}$$

and

$$\Gamma = (1 - v^2/c^2)^{-1/2}. \tag{1.1c}$$

Similarly, the heat reservoir will be described by the 4-vector T^j_R where

$$T^j_R = T_R v^j_R/c = T^*_R(1, \mathbf{v}_R/c), \tag{1.2a}$$

$$T^*_R = \Gamma_R \, T_R, \tag{1.2b}$$

$$\Gamma_R = (1 - v_R^2/c^2)^{-1/2}, \tag{1.2c}$$

where obviously v_R is the 3-velocity of the reservoir, T_R is its temperature in its own rest-frame, and T^*_R is its temperature in the observer's frame. The 4-vector T^j_R must satisfy the reversibility condition given in (4.27a) of I:

$$v_j \, T^j_R = v_j \, T^j = c\,T. \tag{1.3}$$

As in I, the antisymmetric electromagnetic field tensor F^{jk} is expressed in terms of the electric and magnetic field intensities $\mathbf{E}$ and $\mathbf{B}$ as follows:

$$(F^{10},\ F^{20},\ F^{30}) = (E_x,\ E_y,\ E_z) = \mathbf{E}; \tag{1.4a}$$

$$(F^{23},\ F^{31},\ F^{12}) = -\,(B_x,\ B_y,\ B_z) = -\,\mathbf{B}. \tag{1.4b}$$

If A^j is the electromagnetic 4-potential, then

$$F^{jk} = \partial^j A^k - \partial^k A^j. \tag{1.5}$$

In terms of the scalar potential A° and the 3-vector potential $\mathbf{A}$

$$A^j = (A^\circ,\ \mathbf{A}). \tag{1.6}$$

From (1.4) - (1.6) it follows that

$$\mathbf{E} = -\nabla A^\circ - \partial \mathbf{A}/c\,\partial t; \tag{1.7a}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.7b}$$

We shall have occasion to decompose the tensor F^{jk} into the two antisymmetric tensors $\hat{F}^{jk}$ and $\tilde{F}^{jk}$ which are its parallel and normal projections with respect to the 4-velocity v^j. The tensor $\hat{F}^{jk}$ is characterized by the condition

$$\hat{F}^{jk} v_k = F^{jk} v_k . \tag{1.8a}$$

Thus the electromagnetic force calculated using $\hat{F}^{jk}$ is the same as that calculated from F^{jk}. The condition (1.8a) alone does not suffice to specify $\hat{F}^{jk}$ uniquely. It must be augmented by the condition that in the fluid rest frame the space-space components of $\hat{F}^{jk}$ vanish, i.e.

$$(\hat{F}^{23}, \hat{F}^{31}, \hat{F}^{12})_{v=0} = 0 . \tag{1.8b}$$

From (1.8a) it follows that

$$(\hat{F}^{10}, \hat{F}^{20}, \hat{F}^{30})_{v=0} = \mathring{\mathbf{E}}, \tag{1.8c}$$

where $\mathring{\mathbf{E}}$ is the electric field intensity in the fluid rest frame. It is easily verified that $\hat{F}^{jk}$ must have the following form:

$$\hat{F}^{jk} = (F^{j\ell} v_\ell v^k + v^j v_\ell F^{\ell k})/c^2 ; \tag{1.9a}$$

or

$$(\hat{F}^{10}, \hat{F}^{20}, \hat{F}^{30}) = \Gamma \hat{\mathbf{E}} , \tag{1.9b}$$

$$(\hat{F}^{23}, \hat{F}^{31}, \hat{F}^{12}) = \Gamma \hat{\mathbf{E}} \times \mathbf{v}/c, \tag{1.9c}$$

where

$$\hat{\mathbf{E}} = \Gamma [(\mathbf{E} - \mathbf{B} \times \mathbf{v}/c) - (\mathbf{E} \cdot \mathbf{v}) \mathbf{v}/c^2] . \tag{1.9d}$$

Note that $\hat{\mathbf{E}}$ is just the effective electric field intensity that appears in the relativistic form of Ohm's Law.[1]

The antisymmetric tensor $\tilde{F}^{jk}$ is given by

$$\tilde{F}^{jk} = F^{jk} - \hat{F}^{jk} . \tag{1.10a}$$

From (1.4) and (1.9) we find

$$(\tilde{F}^{10}, \tilde{F}^{20}, \tilde{F}^{30}) = \Gamma \tilde{\mathbf{B}} \times \mathbf{v}/c , \tag{1.10b}$$

$$(\tilde{F}^{23}, \tilde{F}^{31}, \tilde{F}^{12}) = -\Gamma \tilde{\mathbf{B}} , \tag{1.10c}$$

where

$$\tilde{\mathbf{B}} = \Gamma [(\mathbf{B} + \mathbf{E} \times \mathbf{v}/c) - (\mathbf{B} \cdot \mathbf{v}) \mathbf{v}/c^2] . \tag{1.10d}$$

In the fluid rest frame we have

$$(\tilde{F}^{10}, \tilde{F}^{20}, \tilde{F}^{30})_{v=0} = 0 , \tag{1.11a}$$

$$(\tilde{F}^{23}, \tilde{F}^{31}, \tilde{F}^{12})_{v=0} = -\mathring{\mathbf{B}} . \tag{1.11b}$$

Thus $\tilde{F}^{jk}$ is the tensor to be associated with effects such as Larmor precession that in the fluid rest frame depend only on $\mathring{\mathbf{B}}$ and not on $\mathring{\mathbf{E}}$. We shall refer to $\hat{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ as the effective electric and magnetic field intensities.

Fluid Rotation Tensor

As a preliminary to deriving the relativistic Larmor theorem in the next section, we shall now discuss the rotation tensor for the fluid. First we note that if $X^j = (0, \mathbf{r})$ is the space-time displacement between two points that are observed simultaneously ($X^0 = c\Delta t = 0$), and if one point is rotating about the other (the origin) with angular velocity $\boldsymbol{\Omega}$ as seen by an observer for whom the origin has the velocity $\mathbf{v}$, then

$$d\mathbf{r}/dt = \boldsymbol{\Omega} \times \mathbf{r} . \tag{1.12}$$

The left side of (1.12) may be put into covariant form by noting that $d\tau = dt/\Gamma$ where $d\tau$ is the proper time interval in the rest frame of the origin. Thus $\Gamma d\mathbf{r}/dt = d\mathbf{r}/d\tau$. If we now introduce an antisymmetric tensor whose space-space components are given by

$$(\Omega^{23}, \Omega^{31}, \Omega^{12}) = \Gamma \boldsymbol{\Omega}, \tag{1.13}$$

then (1.12) can be written in the following covariant form:

$$d X^j/d\tau = \Omega^{jk} X_k . \tag{1.14}$$

The space part of (1.14) is just (1.12). It follows from the antisymmetry of Ω^{jk} that

$$0 = X_j \, dX^j/d\tau = \frac{1}{2} d(X_j X^j)/d\tau = \frac{1}{2} d\, r^2/d\tau . \tag{1.15}$$

This is the necessary condition that the length of the radius vector of a rotating point be constant.

It is well known that the non-relativistic definition of $\boldsymbol{\Omega}$ in the case of a fluid is given by $\boldsymbol{\Omega} = (1/2) \nabla \times \mathbf{v}$. The obvious covariant generalization of this is

$$\Omega^{jk} = -\frac{1}{2} (\partial^j v^k - \partial^k v^j); \tag{1.16a}$$

or

$$(\Omega^{10}, \Omega^{20}, \Omega^{30}) = [\partial(\Gamma \mathbf{v})/\partial t + \nabla(\Gamma c^2)]/2c, \tag{1.16b}$$

$$(\Omega^{23}, \Omega^{31}, \Omega^{12}) = \frac{1}{2} \nabla \times (\Gamma \mathbf{v}) \equiv \Gamma \mathbf{\Omega}. \tag{1.16c}$$

The last equality in (1.16c) constitutes the definition of the angular velocity $\mathbf{\Omega}$ in the relativistic case. The factor Γ has been inserted into this definition in order to give (1.16c) the same form as (1.13).

In the same way that we decomposed F^{jk}, we shall decompose Ω^{jk} into two antisymmetric tensors $\hat{\Omega}^{jk}$ and $\tilde{\Omega}^{jk}$ which are the parallel and normal projections with respect to v^j. The tensor $\hat{\Omega}^{jk}$ is defined by the conditions

$$\hat{\Omega}^{jk} v_k = \Omega^{jk} v_k = \frac{1}{2} d v^j/d\tau \tag{1.17a}$$

and

$$(\hat{\Omega}^{23}, \hat{\Omega}^{31}, \hat{\Omega}^{12})_{v=0} = 0. \tag{1.17b}$$

The last step of (1.17a) follows from (1.16a) above and (1.5), (1.6a) of I. It is easily verified that $\hat{\Omega}^{jk}$ has the following form:

$$\hat{\Omega}^{jk} = (\Omega^{j\ell} v_\ell v^k + v^j v_\ell \Omega^{\ell k})/c^2$$

$$= [(d v^j/d\tau) v^k - (d v^k/d\tau) v^j]/2c^2; \tag{1.18a}$$

or

$$(\hat{\Omega}^{10}, \hat{\Omega}^{20}, \hat{\Omega}^{30}) = (\Gamma^2/2c) d \mathbf{v}/d\tau, \tag{1.18b}$$

$$(\hat{\Omega}^{23}, \hat{\Omega}^{31}, \hat{\Omega}^{12}) = \Gamma \mathbf{\Omega}_T \tag{1.18c}$$

where

$$\mathbf{\Omega}_T = -\ (\Gamma^2/2c^2)\ \mathbf{v} \times D\mathbf{v} \tag{1.18d}$$

where $D = \Gamma^{-1}(d/d\tau)$ is, as defined in (1.6b) of I, the substantial time differentiation operator in the observer's frame. Note that $\mathbf{\Omega}_T$ is just the well-known Thomas precession.[2]

The antisymmetric tensor $\tilde{\Omega}^{jk}$ is given by

$$\tilde{\Omega}^{jk} = \Omega^{jk} - \hat{\Omega}^{jk}. \tag{1.19}$$

From (1.17a) and (1.19)

$$\tilde{\Omega}^{jk}\ v_k = 0. \tag{1.20}$$

Any antisymmetric tensor satisfying the condition (1.20) has only three degrees of freedom and can be written in the form

$$(\tilde{\Omega}^{10}, \tilde{\Omega}^{20}, \tilde{\Omega}^{30}) = (\Gamma/c)\ \mathbf{v} \times \tilde{\mathbf{\Omega}}, \tag{1.21a}$$

$$(\tilde{\Omega}^{23}, \tilde{\Omega}^{31}, \tilde{\Omega}^{12}) \equiv \Gamma\,\tilde{\mathbf{\Omega}}, \tag{1.21b}$$

where (1.21b) constitutes the definition of the 3-vector $\tilde{\mathbf{\Omega}}$ in terms of the space-space components of $\tilde{\Omega}^{jk}$. From (1.19), (1.16c), (1.18c), and (1.21b) it follows that

$$\mathbf{\Omega} = \tilde{\mathbf{\Omega}} + \mathbf{\Omega}_T. \tag{1.22}$$

In the next section we shall see that $\tilde{\mathbf{\Omega}}$ is the fluid rotation that is produced by the magnetic and thermal fields plus the residual rotation that has been retained from the initial conditions of the fluid. Equation (1.22) simply says that in order to arrive at the total rotation Ω, we must also include the Thomas precession Ω_T.

Entropy Force Tensor

It will be necessary to carry out the same kind of decomposition for the entropy force tensor Θ^{jk} which in (5.12) of I was defined as

$$\Theta^{jk} = T_R^j \, \partial^k s - T_R^k \, \partial^j s . \tag{1.23}$$

From (4.26a) of I we have, making the notation change noted in writing (1.1),

$$T_R^j = (T/c) \; (v^j + w^j), \tag{1.24a}$$

or

$$T_R^j = T^* \; [(1 + \mathbf{v} \cdot \mathbf{w}/c^2), \; (\mathbf{v} + \mathbf{w})/c] \tag{1.24b}$$

where $T^* = \Gamma T$ is the temperature in the observer's frame and $\mathbf{v} + \mathbf{w} = \mathbf{v}_R$ is the reservoir 3-velocity. The 3-velocity $\mathbf{w}$ is obviously the drift velocity with respect to the fluid. Using (1.24) and (1.23), we arrive at the following relations:

$$\Theta^{jk} = (T/c) \; [(v^j + w^j) \, \partial^k s - (v^k + w^k) \, \partial^j s] \, ; \tag{1.25a}$$

$$(\Theta^{10}, \Theta^{20}, \Theta^{30}) = T^* \, \nabla s + (T^*/c^2) \; [(\mathbf{v} + \mathbf{w}) \, \partial s/\partial t + (\mathbf{v} \cdot \mathbf{w}) \nabla s] \, ; \tag{1.25b}$$

$$(\Theta^{23}, \Theta^{31}, \Theta^{12}) = (T^* \nabla s) \times (\mathbf{v} + \mathbf{w})/c . \tag{1.25c}$$

The tensor $\hat{\Theta}^{jk}$ is defined by the conditions

$$\hat{\Theta}^{jk} \, v_k = \Theta^{jk} \, v_k = T_R^j \, \dot{s} - c \, T \, \partial^j s \tag{1.26}$$

and

$$(\hat{\Theta}^{23},\ \hat{\Theta}^{31},\ \hat{\Theta}^{12})_{v=0} = 0, \tag{1.27}$$

where the second step of (1.26) follows from (5.13) of I. Thus

$$\hat{\Theta}^{jk} = (\Theta^{j\ell}\, v_\ell\, v^k + v^j\, v_\ell\, \Theta^{\ell k})/c^2 \tag{1.28a}$$

$$= (T/c)\ [(w^j\ \dot{s}/c^2 - \partial^j s)\ v^k - (w^k\ \dot{s}/c^2 - \partial^k s)\ v^j]; \tag{1.28b}$$

$$(\hat{\Theta}^{10},\ \hat{\Theta}^{20},\ \hat{\Theta}^{30}) = \Gamma\,\hat{\mathbf{\Theta}}, \tag{1.28c}$$

$$(\hat{\Theta}^{23},\ \hat{\Theta}^{31},\ \hat{\Theta}^{12}) = \Gamma\,\hat{\mathbf{\Theta}} \times \mathbf{v}/c, \tag{1.28d}$$

where

$$\hat{\mathbf{\Theta}} = T\,\nabla s + (mc^2)^{-1}\ [(m T^*\,\dot{s})\,\mathbf{w} + (m T\,\partial s/\partial t - m T^*\,\dot{s}\,\mathbf{v}\cdot\mathbf{w}/c^2)\,\mathbf{v}]. \tag{1.28e}$$

The antisymmetric tensor $\tilde{\Theta}^{jk}$ is given by

$$\tilde{\Theta}^{jk} = \Theta^{jk} - \hat{\Theta}^{jk}. \tag{1.29}$$

From (1.26) and (1.29) it follows that

$$\tilde{\Theta}^{jk}\, v_k = 0. \tag{1.30}$$

Using (1.25) and (1.28) in (1.29) we find

$$\tilde{\Theta}^{jk} = (T/c)\ [w^j\ (\partial^k s - v^k\,\dot{s}/c^2) - w^k\ (\partial^j s - v^k\,\dot{s}/c^2)]; \tag{1.31a}$$

$$(\tilde{\Theta}^{10},\ \tilde{\Theta}^{20},\ \tilde{\Theta}^{30}) = -\Gamma\,\tilde{\mathbf{\Theta}} \times \mathbf{v}/c, \tag{1.31b}$$

$$(\tilde{\Theta}^{23},\ \tilde{\Theta}^{31},\ \tilde{\Theta}^{12}) = \Gamma\,\tilde{\mathbf{\Theta}}, \tag{1.31c}$$

where

$$\tilde{\boldsymbol{\Theta}} = (T\,\nabla s + T^*\,\dot{s}\,\mathbf{v}/c^2) \times \mathbf{w}/c\,. \tag{1.31d}$$

It is evident that Θ^{jk} can be written in terms of the 3-vector $\hat{\boldsymbol{\Theta}}$ and $\tilde{\boldsymbol{\Theta}}$ as follows:

$$(\Theta^{10},\ \Theta^{20},\ \Theta^{30}) = \Gamma\,(\hat{\boldsymbol{\Theta}} - \tilde{\boldsymbol{\Theta}} \times \mathbf{v}/c)\,, \tag{1.32a}$$

$$(\Theta^{23},\ \Theta^{31},\ \Theta^{12}) = \Gamma\,(\tilde{\boldsymbol{\Theta}} + \hat{\boldsymbol{\Theta}} \times \mathbf{v}/c)\,. \tag{1.32b}$$

Note that in the above relations the relative velocity of the reservoir w^j or $\mathbf{w}$ may be replaced by the thermal force φ^j or $\boldsymbol{\varphi}$ by means of the relation (4.23) of I:

$$\varphi^j = (m\,T\,\dot{s}/c^2)\,w^j\,; \tag{1.33a}$$

$$\boldsymbol{\varphi} = (m\,T\,\dot{s}/c^2)\,\mathbf{w}\,. \tag{1.33b}$$

Gravitation

It would be desirable to include the effects of gravitation to the extent that they are of importance in practical magnetofluid-dynamical problems. For this purpose General Relativity is not needed. A simple flat-space covariant scalar theory will suffice. A variety of such theories have been developed.[3] The first and simplest of these, and the one that we shall employ, is Nordstrom's theory.[4] This theory requires that the rest mass μ of a particle be exponentially dependent on the scalar gravitational potential G, i.e $\mu = m\,\exp\,(G/c^2)$ where m is a constant. The necessity for this dependence follows from the equation of motion (in the absence of all fields but the gravitational one): $d(\mu v^j)/d\tau = \mu\,\partial^j G$. Contraction with v_j gives $d\mu/d\tau = (\mu/c^2)\,dG/d\tau$, and integrating this yields $\mu = m\,\exp\,(G/c^2)$

where m is the integration constant. Note that if the equation of motion had been taken as $d(\mu v^j)/d\tau = m\partial^j G$, where the coupling is given by the constant mass m rather than the variable mass μ, then we would have been led to the linear relation $\mu = m(1 + G/c^2)$ rather than the exponential one given above. The difference between these two relations is of order $m\,(G/c^2)^2$, which for our purposes is completely negligible. Thus we shall adopt the simpler linear relation, although in principle the exponential one is preferable because it is consistent with the Equivalence Principle in that the same mass μ that describes the inertial properties of the particle also describes the strength of its coupling to the gravitational field.

Referring to (5.11) of I, we recall that the variable particle mass $\tilde{m} = m(1 + h/c^2)$ both determined the inertial properties of the particle and also served as a thermal potential function. We may now define a total particle rest mass μ that includes both the gravitational and thermal contributions as follows:

$$\mu = m\,(1 + G/c^2 + h/c^2)\,, \tag{1.34}$$

with the result that

$$\partial^j \mu c^2 = m\,\partial^j G + m\,\partial^j h\,. \tag{1.35}$$

Thus the 4-gradient of the variable particle rest energy μc^2 gives both the gravitational force and the enthalpy-dependent thermal force. The fluid equation of motion including gravitation is just (5.17) of I with $\tilde{m}$ replaced by μ:

$$d\,(\mu v^j)/d\tau = \partial^j\,(\mu c^2) + (m/c)\,\Theta^{jk}\,v_k$$

$$- (\partial_k\,s^{jk})/\rho + (q/c)\,F^{jk}\,v_k\,. \tag{1.36}$$

Because no assumption has been made regarding the equation of state of the fluid, we are not restricting ourselves to the most commonly considered special case of a barotropic fluid.

In the next section we shall find that the forces arising from G and h are represented in the relativistic Larmor theorem by means of the tensor G^{jk} which is defined as follows:

$$G^{jk} = (c/m)\,[(\partial^j \mu)\, v^k - (\partial^k \mu)\, v^j]$$

$$= (\partial^j G + \partial^j h)\, v^k/c - (\partial^k G + \partial^k h)\, v^j/c\,; \qquad (1.37a)$$

$$(G^{10},\ G^{20},\ G^{30}) = -\,\Gamma\,[\nabla(G+h) + \mathbf{v}\ \partial(G+h)/c^2\,\partial t]\,, \qquad (1.37b)$$

$$(G^{23},\ G^{31},\ G^{12}) = -\,\Gamma\,\nabla\,(G+h) \times \mathbf{v}/c\,. \qquad (1.37c)$$

Because $(G^{23},\ G^{31},\ G^{12})_{v=0} = 0$, we have for the tensors $\hat{G}^{jk}$ and $\tilde{G}^{jk}$ that are analogous to the tensors $\hat{F}^{jk}$ and $\tilde{F}^{jk}$ in the case of the electromagnetic field

$$\hat{G}^{jk} = G^{jk} \qquad (1.38a)$$

and

$$\tilde{G}^{jk} = 0\,. \qquad (1.38b)$$

II. LARMOR THEOREM FOR INVISCID FLOW

In this section and the next one we shall neglect the effects of viscosity. Thus we assume that in (1.36)

$$\partial_k S^{jk} = 0 \tag{2.1}$$

where S^{jk} is the viscosity stress tensor.

As a preliminary to rewriting the fluid equation of motion (1.36), we note that the two terms of this equation that involve the variable particle mass μ can be rewritten as follows:

$$d(\mu v^j)/d\tau - \partial^j (\mu c^2) = - [\partial^j (\mu v^k) - \partial^k (\mu v^j)]\, v_k . \tag{2.2}$$

Making use of (1.5), (2.1), and (2.2), we find that (1.36) can be written in the following form:

$$[(\partial^j p^k - \partial^k p^j) + (m/c)\, \Theta^{jk}]\, v_k = 0 \tag{2.3}$$

where p^j is the canonical particle momentum defined by the relation

$$p^j = \mu v^j + (q/c) A^j = (p^0, \mathbf{p}); \tag{2.4a}$$

$$p^0 = (\mu^* c^2 + q A^0/c), \tag{2.4b}$$

$$\mathbf{p} = (\mu^* \mathbf{v} + q \mathbf{A}/c), \tag{2.4c}$$

where $\mu^* = \Gamma\mu$ is the variable particle mass in the observer's frame.

We now introduce the antisymmetric canonical vorticity tensor V^{jk}, defined as follows:

$$V^{jk} = -(\partial^j p^k - \partial^k p^j); \tag{2.5a}$$

$$(V^{10}, V^{20}, V^{30}) = \partial \mathbf{p}/c\,\partial t + \nabla p^0 \equiv \mathbf{F}/c \tag{2.5b}$$

$$(V^{23}, V^{31}, V^{12}) = \nabla \times \mathbf{p} \equiv \mathbf{V}; \tag{2.5c}$$

where (2.5b) and (2.5c) constitute the definitions of the 3-force $\mathbf{F}$ and the 3-vorticity $\mathbf{V}$ in terms of the components of V^{jk}.

We note now that (2.3) implies that

$$V^{jk} = (m/c)\,\Theta^{jk} + 2\mu\,\nu^{jk} \tag{2.6}$$

where ν^{jk} is an antisymmetric tensor that is undetermined except for the condition

$$\nu^{jk} v_k = 0 \tag{2.7}$$

which implies that it can be written in terms of a completely arbitrary 3-vector $\boldsymbol{\nu}$ as follows:

$$(\nu^{10}, \nu^{20}, \nu^{30}) = -\Gamma\,\boldsymbol{\nu} \times \mathbf{v}/c; \tag{2.8a}$$

$$(\nu^{23}, \nu^{31}, \nu^{12}) \equiv \Gamma\,\boldsymbol{\nu}. \tag{2.8b}$$

We shall see that ν^{jk} can be interpreted as the tensor that describes the residual rotation of the fluid, i.e. that part of the total rotation Ω^{jk} that is not produced by the action of the external fields on the fluid, but rather must be regarded as one of the initial starting conditions of the fluid which has been preserved unchanged. (It was to enhance this interpretation that in (2.6) the factor 2μ was inserted in front of ν^{jk}.)

Using (1.32), (2.5), and (2.8), we can rewrite (2.6) as the following two 3-vector equations:

$$\mathbf{F} = \partial \mathbf{p}/\partial t + c \nabla p^0 = m^* (\hat{\boldsymbol{\Theta}} - \tilde{\boldsymbol{\Theta}} \times \mathbf{v}/c) - 2\mu^* \boldsymbol{\nu} \times \mathbf{v} ; \tag{2.9a}$$

$$\mathbf{V} = (m^*/c)(\tilde{\boldsymbol{\Theta}} + \hat{\boldsymbol{\Theta}} \times \mathbf{v}/c) + 2\mu^* \boldsymbol{\nu} . \tag{2.9b}$$

The first of these is an alternative form of the equation of motion. The second expresses the canonical 3-vorticity $\mathbf{V}$ in terms of the residual rotation $\boldsymbol{\nu}$ and the effects of the entropy-dependent forces.

Using (1.16a), (1.37a), and (2.4a) in the definition of V^{jk} given in (2.5a), we arrive at the following relation:

$$V^{jk} = 2\mu\Omega^{jk} - (m/c)\, G^{jk} - (q/c)\, F^{jk} . \tag{2.10}$$

Making use of this to eliminate V^{jk} in (2.6), we arrive at the desired relativistic generalization of the Larmor theorem for inviscid flow of a charged fluid:

$$\Omega^{jk} = (q/2\mu c)\, F^{jk} + (m/2\mu c)\, G^{jk} + (m/2\mu c)\, \Theta^{jk} + \nu^{jk} . \tag{2.11}$$

The first term on the right is the covariant representation of the familiar Larmor precession. The second and third terms are completely analogous precessions that are produced by the μ-dependent and entropy-dependent forces respectively. (By "μ-dependent force" we mean, of course, the sum of the gravitational and enthalpy-dependent forces.) The last term on the right side is completely undetermined except for the condition (2.7). Obviously this represents the fluid rotation that would be present in the absence of any forces, and so must be regarded as an initial starting condition of the fluid. Note that the tensors G^{jk} and Θ^{jk} have been defined in such a way that their coefficients in (2.11) have the same form as that of F^{jk} except that the charge q is replaced by the mass m.

The physical significance of (2.11) becomes most apparent if it is decomposed into one equation in which all the space-time components vanish in the fluid rest-frame, and a second equation in which all the space-space components vanish in this frame. The first of these equations is

$$\tilde{\Omega}^{jk} = (q/2\mu c)\,\tilde{F}^{jk} + (m/2\mu c)\,\tilde{\Theta}^{jk} + \nu^{jk}, \tag{2.12}$$

and the second is

$$\hat{\Omega}^{jk} = (q/2\mu c)\,\hat{F}^{jk} + (m/2\mu c)\,G^{jk} + (m/2\mu c)\,\hat{\Theta}^{jk}. \tag{2.13}$$

Using (1.10c), (1.21b), (1.31c), and (2.8b), we find that the space-space part of (2.12) is equivalent to the following 3-vector statement of the Larmor theorem:

$$\tilde{\mathbf{\Omega}} = -(q/2\mu c)\,\tilde{\mathbf{B}} + (m/2\mu c)\,\tilde{\mathbf{\Theta}} + \mathbf{v} \tag{2.14}$$

where $\mathbf{B}$ is the effective magnetic field defined in (1.10d) and $\tilde{\mathbf{\Theta}}$ is the analogous thermal field defined in (1.31d). Note that the μ-dependent force, which is a potential force, produces no precession in the fluid. As indicated in (1.22), the total rotation $\mathbf{\Omega}$ differs from the $\tilde{\mathbf{\Omega}}$ given above only by the Thomas precession $\mathbf{\Omega}_T$. The 3-vector equation that corresponds to the space-time components of (2.12) is obtained simply by taking the vector cross-product of (2.14) with the fluid 3-velocity $\mathbf{v}$.

Using (1.9b), (1.18b), (1.28c), and (1.37b), we find that the space-time part of (2.13) is equivalent to the following 3-vector equation:

$$\mu^* \, d\mathbf{v}/d\tau = -m\nabla(G+h) + q\hat{\mathbf{E}} + m\hat{\boldsymbol{\Theta}} - \mathbf{v}\,\partial\,[m(G+h)/c^2]/\partial t, \qquad (2.15)$$

where $\hat{\mathbf{E}}$ is the effective electric field intensity defined in (1.9d) and $\hat{\boldsymbol{\Theta}}$ is the analogous entropy force defined in (1.28e). This is an alternative form of the fluid equation of motion that is characterized by the fact that the particle mass μ^*, which includes the speed-dependent mass increase, stands outside the time derivative on the left.

We can cast this equation of motion into a more explicit form by making use of (1.9d), (1.28e), and (5.2) of I:

$$\rho^* \mu^* D\mathbf{v} = -\rho m \nabla G - \nabla \wp + \rho^* q\,[(\mathbf{E} - \mathbf{B} \times \mathbf{v}/c) - (\mathbf{E}\cdot\mathbf{v})\,\mathbf{v}/c^2]$$

$$- \mathbf{v}\,[\rho m\,\partial G/\partial t + \partial\wp/\partial t + (\rho^* T^* m D s/c^2)\,\mathbf{w}\cdot\mathbf{v}]/c^2$$

$$+ (\rho^* T^* m D s/c^2)\,\mathbf{w}. \qquad (2.16)$$

The last term on the right side of (2.16) is a force that arises when heat energy is transferred from the reservoir to the fluid. This energy has a mass associated with it, and the time rate at which heat mass is injected into unit volume of the fluid is $\rho^* T^* mDs/c^2$. Because this mass has the velocity $\mathbf{w}$ relative to the fluid, its absorption produces a drag force $(\rho^* T^* mDs/c^2)\,\mathbf{w}$ that tends to accelerate the fluid in the direction of the reservoir velocity.

As was pointed out in Section IV of I, this drag force performs the work $(\rho^* T^* m D s/c^2)\,\mathbf{w}\cdot\mathbf{v}$ per unit volume of the fluid. Work is also performed on the fluid by increases in the gravitational potential G and the pressure $\mathbb{P}$. (cf. (2.18) of I). These three forms of work, unlike work performed by the electromagnetic field, cause an increase in the rest-mass of the fluid. Such an increase in rest-mass, without the application of a corresponding force to accelerate the newly added mass to the fluid velocity, causes the fluid to decelerate so as to conserve its momentum. This effect results in a drag force in the direction $-\mathbf{v}$. This is the explanation of the next-to-last term on the right side of (2.16).

We have shown that the Larmor theorem (2.14) and the equation of motion (2.16) are parts of a single tensor equation given in (2.11) that may be regarded as the covariant statement of the Larmor theorem. The contraction of (2.11) with the fluid 4-velocity yields the 4-vector equation of motion given in (1.36) (for $\partial_k S^{jk} = 0$). We shall now apply this formalism to derive the relativistic generalization of the Helmholtz equation for inviscid flow of a charged fluid in the presence of electromagnetic and gravitational fields.

III. HELMHOLTZ THEOREM FOR INVISCID FLOW

As a preliminary to deriving the generalized Helmholtz equation, we must first derive an alternative to the expression for **F** given in (2.9a). If we define the 3-vector $\tilde{\Theta}'$ as

$$\tilde{\Theta}' = \tilde{\Theta} + \hat{\Theta} \times \mathbf{v}/c, \tag{3.1}$$

then it follows from (1.25c) and (1.32) that

$$(\Theta^{23},\ \Theta^{31},\ \Theta^{12}) = \Gamma\,\tilde{\Theta}' = \Gamma\,[(T\,\nabla\,s) \times (\mathbf{v} + \mathbf{w})/c]\cdot \tag{3.2a}$$

With this we can derive the following alternative to (1.25b):

$$(\Theta^{10},\ \Theta^{20},\ \Theta^{30}) = -\,\Gamma\,\tilde{\Theta}' \times \mathbf{v}/c + (T\,\nabla\,s)/\Gamma + (T\dot{s}/c^2)(\mathbf{v} + \mathbf{w}). \tag{3.2b}$$

From (2.5), (2.6), (2.8), and (3.2) we find

$$\mathbf{F} = c(V^{10},\ V^{20},\ V^{30}) = \mathbf{v} \times \mathbf{V} + (m\,T\,\nabla s)/\Gamma + (m\,T\,\dot{s}/c^2)(\mathbf{v} + \mathbf{w}) \tag{3.3a}$$

where

$$\mathbf{V} = (V^{23},\ V^{31},\ V^{12}) = (m^*/c)\,\tilde{\Theta}' + 2\mu^*\,\mathbf{v} \tag{3.3b}$$

Equation (3.3a) is the desired expression for **F**.

The covariant vorticity conservation equation is

$$\partial^j\,V^{k\ell} + \partial^k\,V^{\ell j} + \partial^\ell\,V^{jk} = 0, \tag{3.4a}$$

which follows immediately from the form of the definition of V^{jk} given in (2.5a).

(3.4a) is equivalent to the following two equations:

$$\nabla \cdot \mathbf{V} = 0; \tag{3.4b}$$

$$\partial \mathbf{V}/\partial t = \nabla \times \mathbf{F}. \tag{3.4c}$$

Using (3.3a) in (3.4c) we have

$$\partial \mathbf{V}/\partial t = \nabla \times (\mathbf{v} \times \mathbf{V}) + \nabla \times [(m T \nabla s)/\Gamma] + \nabla \times [(m T \dot{s}/c^2)(\mathbf{v} + \mathbf{w})]. \tag{3.5}$$

Substituting (3.4b) and the continuity equation

$$\nabla \cdot \mathbf{v} = -(D \rho^*)/\rho^* \tag{3.6}$$

into the identity

$$\nabla \times (\mathbf{v} \times \mathbf{V}) = -\mathbf{v} \cdot \nabla \mathbf{V} + \mathbf{v}(\nabla \cdot \mathbf{V}) + \mathbf{V} \cdot \nabla \mathbf{v} - \mathbf{V}(\nabla \cdot \mathbf{v}) \tag{3.7}$$

and using the resulting relation in (3.5), we arrive at the generalized Helmholtz equation:

$$D(\mathbf{V}/\rho^*) = (\mathbf{V}/\rho^*) \cdot \nabla \mathbf{v} + (1/\rho^*) \nabla \times [(1 - v^2/c^2)\, \dot{m} T^* \nabla s]$$

$$+ (1/\rho^*) \nabla \times [(m T^* Ds/c^2)(\mathbf{v} + \mathbf{w})], \tag{3.8}$$

where in the last two terms we have made use of the fact that $T/\Gamma = (1 - v^2/c^2)T^*$ and $T \dot{s} = T^* Ds$.

Because

$$\mathbf{V} = \nabla \times \mathbf{p} = \nabla \times (\mu^* \mathbf{v} + q \mathbf{A}/c)$$

$$= \nabla \times (\mu^* \mathbf{v}) + (q/c)\, \mathbf{B} \tag{3.9}$$

where

$$\mu^* = m(1 - v^2/c^2)^{-1/2}(1 + G/c^2 + h/c^2), \tag{3.10}$$

the effects of the electromagnetic, gravitational, and thermal fields are included in (3.8), as well as all relativistic effects.

The last term on the right side of (3.8) arises from the injection of heat into the fluid and vanishes when $Ds = 0$. The next-to-last term vanishes if the specific entropy is everywhere the same. Thus if the flow is isentropic ($Ds = \nabla s = 0$), the relativistic Helmholtz equation for inviscid flow of a charged fluid in the presence of electromagnetic, gravitational, and thermal (enthalpy) fields reduces to the familiar form $D(V/\rho^*) = (V/\rho^*) \cdot \nabla \mathbf{v}$ if the generalized vorticity is defined as in (3.9).[5]

The nonrelativistic limit is obtained by letting c go to infinity (except in the ratio q/c). Then $\rho^* \to \rho$, $T^* \to T$, and (3.8) becomes

$$D[(\nabla \times \mathbf{v} + q\mathbf{B}/mc)/\rho] = [(\nabla \times \mathbf{v} + q\mathbf{B}/mc)/\rho] \cdot \nabla \mathbf{v} + (\nabla T \times \nabla s)/\rho. \tag{3.11}$$

(nonrelativistic)

For $\mathbf{B} = 0$ this is just the well-known Vazsonyi[6] generalization of the Helmholtz equation for the case of nonisentropic flow of an uncharged gas.

If we go to the limit $m \to 0$ in (3.8), we have

$$(m = 0) \qquad D(\mathbf{B}/\rho^*) = (\mathbf{B}/\rho^*) \cdot \nabla \mathbf{v}. \tag{3.12}$$

This is the equation that expresses the fact that the magnetic field is "frozen into" a perfectly conducting fluid which, of course, is what we would have if the charge-carrying particles had zero mass.

Having found a generalization of the Helmholtz equation that comes so close to the simple form $D(\mathbf{V}/\rho^*) = (\mathbf{V}/\rho^*) \cdot \nabla \mathbf{v}$, we now pose the question whether it is possible to modify the definition of the vorticity $\mathbf{V}$ in such a way that this simple form of the equation will hold exactly. In the next section it will be shown that this is indeed possible, even when viscosity must be taken into account. The procedure consists of lumping together the entropy-dependent forces and the viscous forces and describing them in terms of a 4-vector function a^j whose dynamical effects are completely analogous to those of the electromagnetic 4-potential A^j. When the definition of the canonical particle momentum is modified to include a contribution from a^j, we shall find that the vorticity, defined as the curl of this generalized canonical momentum, satisfies the simple form of Helmholtz's equation exactly.

IV. GENERALIZED HELMHOLTZ THEOREM

Using (5.13) of I in (1.36) above, the fluid equation of motion including viscosity ($\partial_k S^{jk} \neq 0$) becomes

$$d(\mu v^j)/d\tau = \partial^j(\mu c^2) + (q/c)F^{jk} v_k + \pi^j - mT\partial^j s + \eta^j, \tag{4.1}$$

where π^j is the time rate of energy-momentum injection resulting from heat injection which, using the equations (2.20) and (5.13) of I, can be written

$$\pi^j = -(\partial_k Q^{jk})/\rho = (m/c)\,\Theta^{jk} v_k + m\,T\,\partial^j s. \tag{4.2}$$

$$\eta^j = -(\partial_k S^{jk})/\rho \tag{4.3}$$

is the viscous force. The symmetric tensors Q^{jk} and S^{jk} are, of course, the heat and viscous stress contributions respectively to the fluid stress-energy tensor.

Using (1.5) and (2.2), we can cast (4.1) into the following form:

$$-[\partial^j(\mu v^k + qA^k/c) - \partial^k(\mu v^j + q\,A^j/c)]\,v_k = \pi^j - m\,T\,\partial^j s + \eta^j. \tag{4.4}$$

It is obvious that the contraction of the left side of this equation with v_j vanishes. Thus we conclude that

$$v_j(\pi^j - m\,T\,\partial^j s + \eta^j) = 0. \tag{4.5}$$

Because the force (π^j - $m\,T\,\partial^j s + \eta^j$) must satisfy this orthogonality condition, it can be expressed as the contraction of a suitably chosen antisymmetric tensor with v_k. Thus we define the antisymmetric tensor f^{jk} by means of the following relation:

$$(m/c)\, f^{jk}\, v_k = \pi^j - m T\, \partial^j s + \eta^j. \tag{4.6a}$$

Because

$$f^{jk} = - f^{kj}, \tag{4.6b}$$

it follows that the orthogonality condition (4.5) is automatically satisfied. Referring to (4.2) and (4.3), it is evident that the defining equation for f^{jk} can also be written in either of the following forms:

$$f^{jk} v_k = \Theta^{jk} v_k + (c/m)\, \eta^j ; \tag{4.6c}$$

or

$$f^{jk} v_k = - c\, [T \partial^j s + (1/\rho m)\, \partial_k (Q^{jk} + S^{jk})]. \tag{4.6d}$$

Because of the analogy between f^{jk} and the electromagnetic field tensor F^{jk}, the space-time components of f^{jk} will be designated as $\mathbf{e}$ and the space-space components as $\mathbf{b}$:

$$(f^{10},\ f^{20},\ f^{30}) = \mathbf{e} ; \tag{4.7a}$$

$$(f^{23},\ f^{31},\ f^{12}) = - \mathbf{b}. \tag{4.7b}$$

The equation (4.6a) specifies only $\overset{\circ}{\mathbf{e}}$, the value of $\mathbf{e}$ in the fluid rest frame. It says nothing about $\overset{\circ}{\mathbf{b}}$. We can remove this indeterminancy by requiring that f^{jk} be the curl of a 4-vector a^j:

$$f^{jk} = \partial^j a^k - \partial^k a^j. \tag{4.8}$$

Inasmuch as the addition to a^j of the 4-gradient of an arbitrary scalar function has no effect on f^{jk}, the 4-vector a^j has only three significant degrees of freedom, which corresponds to the fact that the 4-force $(\pi^j - mT\partial^j s + \eta^j)$ has three degrees of freedom. (One degree of freedom is removed by the orthogonality requirement (4.5).) The equation of motion for a^j follows from (4.6a) and (4.8):

$$da^j/d\tau = v_k \partial^j a^k - (c/m)(\pi^j - mT\partial^j s + \eta^j). \tag{4.9}$$

By introducing f^{jk} and a^j we have placed the handling of the viscous and entropy-dependent forces on the same footing as the handling of the electromagnetic forces. In the case of the latter, however, the appropriate field equations are Maxwells equations, whereas in the case of the former, the "field equations" are given by either (4.6) or (4.9). This point of view will be somewhat more fully developed in the following paper.

Our present purpose, however, is to derive the generalized Helmholtz equation. To this end we note that if we substitute (4.6a) into (4.4), the resulting equation of motion is just the contraction of v_k with the following equation:

$$V^{jk} + (m/c)f^{jk} = 2\mu\omega^{jk}, \tag{4.10}$$

where V^{jk} is the canonical vorticity defined in (2.5), and ω^{jk} is an arbitrary antisymmetric tensor that satisfies the orthogonality condition

$$\omega^{jk} v_k = 0 \tag{4.11}$$

which implies that ω^{jk} has the form

$$(\omega^{10}, \omega^{20}, \omega^{30}) = -\Gamma \boldsymbol{\omega} \times \mathbf{v}/c, \tag{4.12a}$$

$$(\omega^{23},\ \omega^{31},\ \omega^{12}) \equiv \Gamma\boldsymbol{\omega} \tag{4.12b}$$

where (4.12b) is the definition of the 3-vector $\boldsymbol{\omega}$ in terms of the components of ω^{jk}.

The tensor $2\mu\omega^{jk}$ will play the role of a generalized vorticity, which we shall call the <u>intrinsic</u> vorticity. It can be written in the form

$$2\mu\omega^{jk} = (\partial^j \not{p}^k - \partial^k \not{p}^j) \tag{4.13}$$

where

$$\not{p}^j = \mu v^j + (q/c)\,A^j + (m/c)\,a^j \tag{4.14}$$

is the generalized canonical particle momentum that includes the contribution $(m/c)\,a^j$ that arises from the viscous and entropy-dependent forces.

Substituting (2.10) into (4.10), we arrive at the generalized Larmor theorem:

$$\Omega^{jk} = (q/2\mu c)F^{jk} + (m/2\mu c)\,G^{jk} + (m/2\mu c)\,f^{jk} + \omega^{jk}. \tag{4.15}$$

A comparison of this with (2.11) shows that

$$(m/2\mu c)(\Theta^{jk} - f^{jk}) = \omega^{jk} - \nu^{jk}. \tag{4.16}$$

Contracting this with v_k and using (2.7) and (4.11), we have

$$(\Theta^{jk} - f^{jk})v_k = 0. \tag{4.17}$$

Thus f^{jk} differs from Θ^{jk} at most by a tensor that is orthogonal to v_k.

The tensor ν^{jk} was interpreted as the residual rotation, i.e. the rotation not produced by external forces, but rather associated with the initial conditions of the fluid. From (4.15) it is evident that a similar interpretation may be assigned to ω^{jk}. To distinguish it from ν^{jk}, we shall call ω^{jk} the intrinsic (rather than the residual) rotation. Of these two tensors, which in general are not equal, ω^{jk} is the more fundamental. This follows from the fact that it is the flux of $\boldsymbol{\omega}$ (or more exactly, the flux of the intrinsic vorticity $2\mu^*\boldsymbol{\omega}$) and not that of $\boldsymbol{\nu}$ that in general is conserved. We shall now prove this.

From (4.13) it follows directly that

$$\partial^j(\mu\,\omega^{k\ell}) + \partial^k(\mu\,\omega^{\ell j}) + \partial^\ell(\mu\,\omega^{jk}) = 0, \tag{4.18a}$$

which is equivalent to the following two equations:

$$\nabla\cdot(\mu^*\boldsymbol{\omega}) = 0, \tag{4.18b}$$

$$\partial(\mu^*\boldsymbol{\omega})/\partial t = \nabla\times(\mathbf{v}\times\mu^*\boldsymbol{\omega}), \tag{4.18c}$$

where use has been made of (4.12). It is well-known[7] that these two equations imply the validity of the following conservation law:

$$D\int_S \mu^*\boldsymbol{\omega}\cdot d\mathbf{S} = 0 \tag{4.19}$$

where $D = (\partial/\partial t) + \mathbf{v}\cdot\nabla$ is the substantial time differentiation operator, and the integral is taken over any surface S which moves with the fluid. Thus the flux of the intrinsic vorticity $2\mu^*\boldsymbol{\omega}$ passing through any closed loop that moves with the fluid never diffuses out of this enclosed area. It is easily verified that this would be true for the $2\mu^*\boldsymbol{\nu}$ flux only if Θ^{jk}, like f^{jk}, were expressible as the curl of a 4-vector.

Because (4.18c) lacks the two entropy-dependent terms that appear in (3.5), when we duplicate the steps that led from (3.5) to (3.8) starting now from (4.18c), we arrive at the following simple form for the generalized Helmholtz equation:

$$D(\mu^* \boldsymbol{\omega}/\rho^*) = (\mu^* \boldsymbol{\omega}/\rho^*) \cdot \nabla \mathbf{v}. \tag{4.20}$$

(The ratio μ^*/ρ^* could of course also be written μ/ρ, but it seems more appropriate to refer everything to the observer's frame.)

It is possible to give this generalized Helmholtz equation an interesting microscopic interpretation which will now be discussed.

Microscopic Interpretation of Helmholtz Equation

We may take the point of view that the mass μ^* of each of the ρ^* particles that occupy unit volume of the fluid is uniformly distributed over a volume $\mathcal{V}_1 = 1/\rho^*$ which for the sake of simplicity we may consider to have the form of a right cylinder of height ℓ and cross-sectional area $\mathcal{a}$. We assume that at some given instant of time the axes of the particle-cylinders are all parallel to the local value of the intrinsic rotation ω, and that the cylinders are rotating about their axes with the angular velocity ω. The total angular momentum of any particle may be regarded as the sum of its external and internal parts. The external part is just the angular momentum about some specified origin of a point mass μ^* located at the center of the particle-cylinder. It thus depends only on the position and displacement velocity of the cylinder, but not on its rotation about its own axis. The internal angular momentum σ is just the product of the angular velocity $\boldsymbol{\omega}$ of the cylinder and its moment of inertia which is $\mu^* a/2\pi = \mu^*/(2\pi \ell \rho^*)$:

$$\boldsymbol{\sigma} = \mu^* \boldsymbol{\omega}/(2\pi \ell \rho^*) . \tag{4.21}$$

We further stipulate that the ends of each particle-cylinder be fixed in the fluid. Thus if $\boldsymbol{\ell}$ is the vector connecting the two ends of the cylinder, it must obey the well-known equation[8] for the time dependence of the vector displacement between two neighboring fluid particles:

$$D\boldsymbol{\ell} = \boldsymbol{\ell} \cdot \nabla \mathbf{v} . \tag{4.22}$$

But this equation has the same form as (4.20). This implies that

$$\mu^* \boldsymbol{\omega}/\rho^* = K\boldsymbol{\ell} \tag{4.23}$$

where K is a constant along the trajectory of any given particle, i.e. $DK = 0$. Thus if at any instant of time the axis of a particle-cylinder is parallel to the local value of $\boldsymbol{\omega}$, then as the cylinder is swept along with the fluid it will find that its orientation is always such that its axis is parallel to the local value of $\boldsymbol{\omega}$. Although the <u>direction</u> of $\boldsymbol{\sigma}$ changes in general as the cylinder is carried along with the fluid, its <u>magnitude</u> $|\boldsymbol{\sigma}|$ is constant. This follows from (4.23):

$$\boldsymbol{\sigma} = \mu^* \boldsymbol{\omega}/(2\pi \ell \rho^*) = (K/2\pi)\, \boldsymbol{\ell}_1 . \tag{4.24}$$

where $\boldsymbol{\ell}_1$ is the unit vector along the cylinder axis. Because $DK = 0$, we have then

$$D|\boldsymbol{\sigma}| = D(K/2\pi) = 0 . \tag{4.25}$$

The behavior of a particle-cylinder whose two ends are fixed to the fluid points A and B is illustrated in Figure 1. As the particle-cylinder is carried along with the fluid, it changes (in general) its size and shape, and hence its moment of inertia, but the magnitude of its internal angular momentum $\sigma = |\boldsymbol{\sigma}|$ remains constant, which means that the moment of inertia must vary inversely with $|\boldsymbol{\omega}|$.

Note that the internal angular velocity of the particle-cylinders is the intrinsic angular velocity $\boldsymbol{\omega}$ and not the fluid rotation $\boldsymbol{\Omega}$. These two differ by the precession produced by the external fields. We conclude then that the external fields are unable to apply a torque about the axis of any given particle-cylinder. Because of this its angular momentum $\boldsymbol{\sigma}$ is conserved except for the fact that, because the cylinder is embedded in the fluid, the axis of the cylinder, and hence the direction of $\boldsymbol{\sigma}$, changes with time. Thus, if the particle-cylinders do not have any internal angular momentum, there is no way in which they can acquire it. This, of course, corresponds to the well-known implication of the Helmholtz equation that, if the intrinsic vorticity $2\mu^*\boldsymbol{\omega}$ vanishes at any point on a particle trajectory, it must be zero along the entire trajectory.

It would be of interest to derive the equation of motion of $\boldsymbol{\sigma}$. From (4.20) and (4.21)

$$D\boldsymbol{\sigma} = \boldsymbol{\sigma} \cdot \nabla \mathbf{v} - \boldsymbol{\sigma}(D\ell)/\ell. \tag{4.26}$$

Using (4.22) we find

$$(D\ell)/\ell = \ell^{-2}(D\boldsymbol{\ell}) \cdot \boldsymbol{\ell} = (\boldsymbol{\ell} \cdot \nabla \mathbf{v}) \cdot \boldsymbol{\ell}/\ell^2$$

$$= (\boldsymbol{\sigma} \cdot \nabla \mathbf{v}) \cdot \boldsymbol{\sigma}/\sigma^2. \tag{4.27}$$

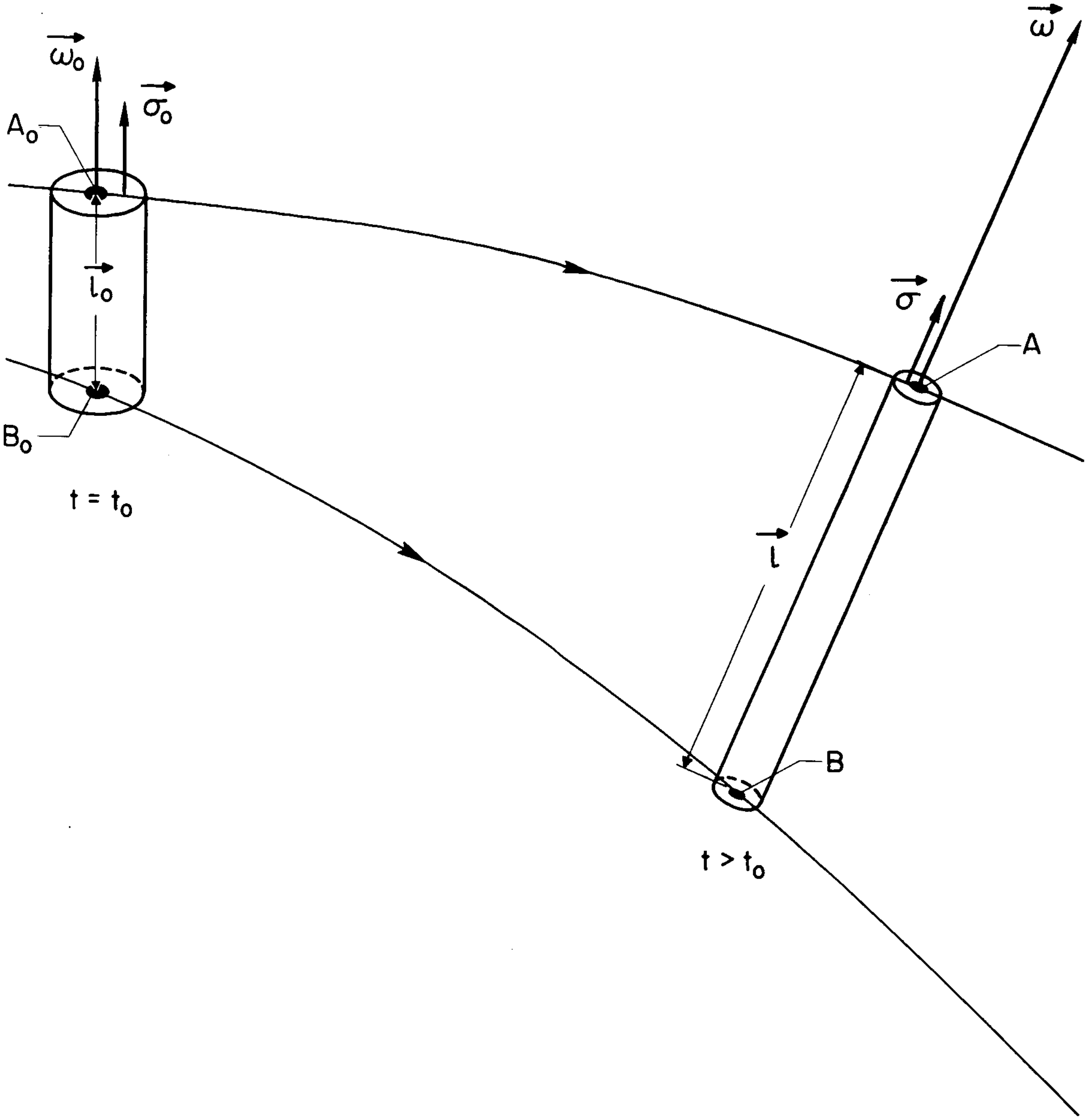

Figure 1—Behavior of a particle-cylinder being swept along with the fluid.

Thus (4.26) becomes

$$D\boldsymbol{\sigma} = \boldsymbol{\sigma}\cdot\nabla\mathbf{v} - \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\nabla\mathbf{v})\cdot\boldsymbol{\sigma}/\sigma^2, \tag{4.28a}$$

or

$$D\boldsymbol{\sigma} = \boldsymbol{\sigma}\times[(\boldsymbol{\sigma}\cdot\nabla\mathbf{v})\times\boldsymbol{\sigma}]/\sigma^2. \tag{4.28b}$$

The second of these two equations shows that the motion of $\boldsymbol{\sigma}$ may be regarded as a precession with the $\boldsymbol{\sigma}$-dependent angular velocity $-(\boldsymbol{\sigma}\cdot\nabla\mathbf{v})\times\boldsymbol{\sigma}/\sigma^2$.

The dependence of the motion of $\boldsymbol{\sigma}$ on the fluid velocity can be made more explicit by rewriting (4.28a) in the following form:

$$D\boldsymbol{\sigma} = [\mathfrak{D}\cdot\boldsymbol{\sigma} - \boldsymbol{\sigma}(\boldsymbol{\sigma}\cdot\mathfrak{D}\cdot\boldsymbol{\sigma})/\sigma^2] + \frac{1}{2}(\nabla\times\mathbf{v})\times\boldsymbol{\sigma}, \tag{4.29a}$$

where $\mathfrak{D}$ is the symmetric fluid distortion dyadic given by

$$\mathfrak{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})_c], \tag{4.29b}$$

where $(\nabla\mathbf{v})_c$ is the conjugate (or transpose) of $\nabla\mathbf{v}$. The second term on the right side of (4.29a) is simply a precession of angular velocity $\frac{1}{2}(\nabla\times\mathbf{v})$, which is the nonrelativistic fluid rotation. Thus this term is simply the precession of the internal angular momentum that results because the axis of the particle-cylinder is imbedded in the fluid and must rotate with it. The first term on the right side of (4.29a) describes the change in the orientation of the cylinder axis (and hence of $\boldsymbol{\sigma}$) that results from fluid distortion.

The precession $\frac{1}{2}\nabla\times\mathbf{v}$ may also be expressed in terms of the external fields. Using (1.4b), (1.16c), (1.37c), (4.7b), and (4.12b) in (4.15) we find

$$\frac{1}{2}\nabla\times\mathbf{v} = -(2\mu^* c)^{-1}\ \{q\mathbf{B} + [m\nabla(G+h) + \mu\nabla\Gamma c^2]\times\mathbf{v}/c + m\mathbf{b}\} + \boldsymbol{\omega}. \tag{4.30}$$

Because $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ are parallel, $\boldsymbol{\omega}\times\boldsymbol{\sigma} = 0$. Thus the precession velocity of $\boldsymbol{\sigma}$ in (4.29a) is given by the first term on the right side of (4.30). The middle term inside the braces is a relativistic precession which results from the gravitational, thermal, and Bernoulli forces, and which vanishes in the rest frame of the fluid.

Equation (4.21) can be used to estimate the magnitude σ of the internal angular momentum, if we assume that $\ell \sim \mho_1^{1/3} = (\rho^*)^{-1/3}$. Taking μ^* to be of the order of the proton mass, we find that for laboratory and astrophysical problems

$$0 \le \sigma \lesssim 10^{-36}\ \text{erg-sec}. \tag{4.31}$$

It is interesting to note that if we were to take a classical hydrodynamical view of a large nucleus, assuming that $\omega \sim 0.1\ c/R$ where the nuclear radius R is of the order of 10^{-13} cm, then we would find that $\sigma \sim \hbar \approx 10^{-27}$ erg-sec, where $\hbar$ is Planck's constant. The angular momenta that one encounters in nuclear and atomic problems are, of course, of the order of magnitude of $\hbar$. Thus the internal angular momentum σ that we encounter in classical problems is much smaller than the typical magnitude of the angular momentum per particle that we encounter in atomic and nuclear problems, and would approach this magnitude only if we were to attempt to apply the classical formalism to problems of atomic and nuclear dimensions.

V. CONCLUSIONS

The formalism based on the description of the viscous and entropy-dependent forces in terms of a 4-potential a^j that is analogous in its dynamical effects to the electromagnetic 4-potential has two advantages over the formalism based on the entropy force tensor Θ^{jk}: First, the 4-potential a^j can be used to describe the viscous as well as the entropy-dependent forces. Second, the intrinsic vorticity $2\mu^*\boldsymbol{\omega}$ in the a^j-formalism is more fundamental than the canonical vorticity $\nabla\times(\mu^*\mathbf{v} + q\mathbf{A}/c)$ that enters into the Θ^{jk}-formalism in that the Helmholtz equation in the former case yields an exact conservation law for the $2\mu^*\boldsymbol{\omega}$ flux whereas the canonical vorticity flux in the latter case is not in general conserved. The most interesting feature of the a^j-formalism, however, is that it allows the fluid equation of motion to be cast into the form of a generalized Hamilton-Jacobi equation, whose derivation is given in the following paper.

REFERENCES

1. See, for example, A. Sommerfeld, Electrodynamics (Academic Press, New York, 1952), p. 283, eq. 9.

2. L. H. Thomas, Nature 117, 514 (1926); Phil. Mag. (7)3, 1 (1927). The derivation of the Thomas precession can be found in many textbooks; e.g. C. Møller, The Theory of Relativity (Oxford, 1952), p. 54-56; or J. D. Jackson, Classical Electrodynamics (Wiley, New York, 1962), p. 367.

The familiar form of the Thomas precession, which is derived in the above references, is $\Omega_T = -[(\Gamma/c)^2/(\Gamma+1)]\ \mathbf{v} \times \mathrm{D}\mathbf{v}$, which differs from (1.18d) by the factor $(1/2)(\Gamma+1) \approx 1+(1/4)v^2/c^2$. It is not clear whether any physical significance can be attached to this difference, which in any case is quantitatively negligible in most situations of physical interest. The intricacy of the question is indicated in the Phil. Mag. paper of Thomas that is cited above. In particular, his equation (9.7) in the note at the end of this paper, in which he discusses a possible alternative formulation of the problem, indicates that in such a formulation the tensor $\hat{\Omega}^{jk}$ defined in (1.18) above might be the natural covariant form of the Thomas precession. In this connection see also J. Frenkel, Z. Physik 37, 243 (1926).

In any case, a complete theory yields only the magnitude of the total precession, which is the sum of the Thomas precession and the precession produced by external fields acting on the fluid. Because the Thomas precession also depends on the external fields, the way in which this total precession is divided between the two types of precession is a somewhat arbitrary and subjective matter. In the present paper, the division that has been made is indicated by the relation $\Omega = \tilde{\Omega} + \Omega_T$ given in (1.22), where $\tilde{\Omega}$ is expressed in terms of the external fields in equation (2.14) of Section II. This way of dividing the total precession Ω into externally produced precession $\tilde{\Omega}$ and Thomas precession Ω_T seems to be the most natural one for the case of relativistic charged fluid flow.

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4. G. Norström, Phys. Z. 13, 1126 (1912).

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6. A. Vazsonyi, Quart. Appl. Math. 3, 29 (1945). Vazsonyi's derivation is also given in a review article by J. Serrin, Encyclopedia of Physics (Springer, Berlin, 1959), Vol. VIII/1, p. 189.

7. See, for example, M. Abraham and R. Becker, The Classical Theory of Electricity and Magnetism (Hafner, New York, 1951), 2nd ed., p. 39-40.

8. See, for example, L. Brand, Vector and Tensor Analysis (Wiley, New York, 1947), §123.